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Information-entropy and the space of decoherence functions in generalised quantum theory

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Abstract

In standard quantum theory, the ideas of information-entropy and of pure states are closely linked. States are represented by density matrices ρ on a Hilbert space and the information-entropy $-\mathrm{tr}(\rho \log \rho)$ is minimised on pure states (pure states are the vertices of the boundary of the convex set of states). The space of decoherence functions in the consistent histories approach to generalised quantum theory is also a convex set. However, by showing that every decoherence function can be written as a convex combination of two other

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decoherence functions we demonstrate that there are no ‘pure’ decoherence functions.

The main content of the paper is a new notion of information-entropy in generalised quantum mechanics which is applicable in contexts in which there is no *a priori* notion of time. Information-entropy is defined first on consistent sets and then we show that it decreases upon refinement of the consistent set. This information-entropy suggests an intrinsic way of giving a consistent set selection criterion.

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I. INTRODUCTION

A particularly attractive feature of the consistent histories programme, as developed by Gell-Mann and Hartle [1–7] following pioneering work by Griffiths [8] and Omnès [9–14] is that it offers a framework for quantum theory in which time potentially plays a subsidiary rôle[1]. The central idea of the scheme is that under certain *consistency* conditions it is possible to assign probabilities to generalised histories of a system. In normal quantum theory such histories are represented by time-ordered strings of propositions; however the scheme allows for much more general histories in which there is no *a priori* notion of time ordering. These generalised histories are expected to play a key rôle in application of the formalism to quantum gravity.

In the generalised version of the history scheme that we have developed [18–20] the central mathematical ingredients are a set of histories $\mathcal{UP}$ (or, more accurately, the set of *propositions* about histories) and an associated set of decoherence functions $\mathcal{D}$, with the pair $(\mathcal{UP}, \mathcal{D})$ being regarded as the analogue in the history theory of the pair $(\mathcal{L}, \mathcal{S})$ in standard quantum theory where $\mathcal{L}$ is the lattice of propositions and $\mathcal{S}$ is the space of states on $\mathcal{L}$.

In this paper we address two related issues: we investigate the structure of the convex set of decoherence functions and we suggest a new definition of information-entropy for decoherence functions. The analogues of these ideas in standard quantum theory are simply related by the fact that the information-entropy $I_{single-time} = -\mathrm{tr}(\rho \log \rho)$ is minimised on the vertices of the boundary of the convex set of density matrices ρ; these vertices are the 'pure' density matrices corresponding to pure states in the Hilbert space. As we shall show, although convex, the space of decoherence functions has a very different structure and there are no 'pure' decoherence functions. For this and other reasons we need a rather different approach to the notion of information-entropy in generalised quantum theory.

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[1]For more recent developments in the consistent histories programme by these authors see, for example, [15–17].

Several other authors [21–24] have considered aspects of information theory in the context of the consistent histories approach. In particular, in a very interesting paper that partly motivated our work, Hartle [21] proposed a definition of information that we describe in section 3. We feel however that our alternative definition has certain advantages over that given in [21]: in particular, it is more straightforward.

II. THE CONVEX SET OF DECOHERENCE FUNCTIONS

A. 'Pure' decoherence functions

In [18,19] we described how the space $\mathcal{UP}$ encodes the generalised quantum temporal logic of the propositions. As explained in [18,19], there are compelling reasons for postulating that the natural mathematical structure on $\mathcal{UP}$ is that of an *orthoalgebra* [25], with the three orthoalgebra operations $\oplus$, $\neg$, and $<$ corresponding respectively to the disjoint sum, negation, and coarse-graining operations invoked by Gell-Mann and Hartle. One example of an orthoalgebra is the lattice of projection operators on a Hibert space. In this case, the operation $\oplus$ is defined on disjoint pairs of projectors P, Q with $P \oplus Q = P \vee Q$ where, as usual, $P \vee Q$ denotes the projector onto the linear span of the subspaces onto which P and Q project. In the example of a lattice (which is a special type of orthoalgebra), $\vee$ is defined on all projectors, not only on pairs that are disjoint.

Throughout this paper we shall be dealing with the case where the orthoalgebra of propositions is the space of projectors on a Hilbert space $\mathcal{V}$ which, for the sake of simplicity, we shall take to be finite dimensional. This Hilbert space may arise from having propositions at n time points, in which case $\mathcal{V} = \otimes^n \mathcal{H}$ (see below), but it need not do so. A crucial ingredient in our construction of the information-entropy will be the *dimension* of a proposition, defined to be the dimension of the projector that represents the proposition on $\mathcal{V}$.

The properties of the decoherence function $d : \mathcal{UP} \times \mathcal{UP} \to \mathbb{C}$ are

1. *Hermiticity*: $d(\alpha, \beta) = d(\beta, \alpha)^*$ for all $\alpha, \beta \in \mathcal{UP}$.

2. *Positivity*: $d(\alpha, \alpha) \geq 0$ for all $\alpha \in \mathcal{UP}$.

3. *Additivity*: if α and β are disjoint then, for all γ, $d(\alpha \oplus \beta, \gamma) = d(\alpha, \gamma) + d(\beta, \gamma)$.

4. *Normalisation*: $d(1, 1) = 1$.

One important motivation for our framework is the fact that discrete-time histories in quantum theory can indeed be given the structure of an orthoalgebra. The key idea is that an n-time, homogeneous history proposition $(\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n})$ can be associated with the operator $\alpha_{t_1} \otimes \alpha_{t_2} \otimes \cdots \otimes \alpha_{t_n}$ which is a genuine *projection* operator on the n-fold tensor product $\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2} \otimes \cdots \otimes \mathcal{H}_{t_n}$ of n-copies of the Hilbert-space $\mathcal{H}$ on which the canonical theory is defined [18,19].

It may be noted that if d_1 and d_2 are decoherence functions then so is

$$d_{(\lambda)} := \lambda d_1 + (1 - \lambda) d_2 \tag{1}$$

where λ is a real constant $0 \leq \lambda \leq 1$. Thus the space of decoherence functions is a convex set.

What we have said so far looks very similar to the situation in standard single-time quantum theory where—by the use of Gleason's theorem—a state may be characterised by a positive self-adjoint operator with trace 1 (*i.e.,* a density matrix) on the Hilbert space. The probability, $\mathrm{Prob}(P; \rho)$, that the proposition represented by the projection operator P is true if the system is in the state ρ and a suitable measurement is made is

$$\mathrm{Prob}(P; \rho) = \mathrm{tr}(P\rho). \tag{2}$$

The space of density matrices is also convex in the sense that

$$\rho_{(\lambda)} := \lambda \rho_1 + (1 - \lambda) \rho_2 \tag{3}$$

is a state if ρ_1 and ρ_2 are states and $0 \leq \lambda \leq 1$. In standard single-time quantum theory a state is said to be *pure* if it cannot be written in the form (3) with $\rho_1 \neq \rho_2$. Pure states play

an important rôle since, in this case, the probabilities (2) cannot be interpreted as arising from a stochastic mixture.

Since a state ρ is a positive self-adjoint operator, the spectral theorem shows that it can be written as

$$\rho = \sum_i r_i P_i \tag{4}$$

where $0 \leq r_i \leq 1$ are the eigenvalues of ρ and P_i are the projectors onto the associated eigenspaces. This shows that unless all the r_i are zero except one, the state is certainly impure. Furthermore, it can be shown [26] that pure states are of the form

$$\rho = P, \tag{5}$$

where P is a projection operator onto a one-dimensional subspace.

In generalised history quantum theory, although the set of decoherence functions is convex, in other respects the situation is quite different from that in standard quantum theory. One may attempt to define a pure decoherence function d as one that cannot be written in the form

$$d = \lambda d_1 + (1-\lambda) d_2 \tag{6}$$

with $d_1 \neq d_2$. However we now show that there are no such decoherence functions.

Firstly, let us recall [20] that we have characterised all decoherence functions in the case that $\mathcal{UP}$ is the lattice of projectors on a finite dimensional Hilbert space as follows [2].

Decoherence functions are in one-to-one correspondence with 'decoherence operators' X on $\mathcal{V} \otimes \mathcal{V}$ according to the rule

$$d(\alpha, \beta) = tr_{\mathcal{V}\otimes\mathcal{V}}(\alpha \otimes \beta X) \tag{7}$$

where the decoherence operator X satisfies

[2] Generalisations of this result have been given in [27,28].

1. $MXM = X^\dagger$ *where* $M(u \otimes v) := v \otimes u$;

2. $tr_{\mathcal{V}\otimes\mathcal{V}}(\alpha \otimes \alpha X) \geq 0$;

3. $tr_{\mathcal{V}\otimes\mathcal{V}}(X) = 1$.

It should be noted that X need not be a positive operator. Indeed, in [19] we found examples of decoherence functions in standard quantum theory where $d(\alpha, \alpha) > d(\beta, \beta)$ for two histories α and β for which $\alpha \leq \beta$, and we also found decoherence functions and histories γ for which $d(\gamma, \gamma) > 1$.

Now consider an operator on $\mathcal{V} \otimes \mathcal{V}$ of the following form

$$Y = i(s_1 \otimes s_2 - s_2 \otimes s_1) \tag{8}$$

for any self-adjoint operators $s_1,\ s_2$ on $\mathcal{V}$. It may be seen that Y satisfies

1. $MYM = Y^\dagger$ where M is the interchange operator given above;

2. $\mathrm{tr}_{\mathcal{V}\otimes\mathcal{V}}(\alpha \otimes \alpha Y) = 0 \quad \forall\alpha$; in particular $\mathrm{tr}_{\mathcal{V}\otimes\mathcal{V}}(Y) = \mathrm{tr}_{\mathcal{V}\otimes\mathcal{V}}(1 \otimes 1\ Y) = 0$.

Given any decoherence operator X_d associated with a decoherence function d, let us define two new operators X_d^+ and X_d^- by

$$X_d^+ = X_d + Y \quad \text{and} \quad X_d^- = X_d - Y. \tag{9}$$

Then X_d^+ and X_d^- are also decoherence operators, as may easily be checked.

Now consider the identity

$$X_d \equiv \frac{1}{2}(X_d + Y) + \frac{1}{2}(X_d - Y) = \frac{1}{2}X_d^+ + \frac{1}{2}X_d^-. \tag{10}$$

It is clear that if d^+ and d^- denote the decoherence functions that are associated with the decoherence operators X^+ and X^- respectively then

$$d = \frac{1}{2}d^+ + \frac{1}{2}d^- \tag{11}$$

and hence d is impure. Thus there are no 'pure' decoherence functions[3].

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[3] Other aspects of the structure of the space of decoherence functions have been considered in [29].

B. Pure decoherence functions with respect to a window

Whilst there are no pure decoherence functions in general, it is possible to discuss a notion of purity of a decoherence function in the context of a fixed consistent set. In general we shall refer to an exclusive and exhaustive set of propositions (*i.e.* a resolution of the identity in the orthoalgebra $\mathcal{UP}$) as a *window* $W = \{\alpha_i\}$ [4].

Firstly, we define two decoherence functions, d_1 and d_2 to be W-*equivalent* if

1. W is a consistent set with respect to both d_1 and d_2 (*i.e.,* $d_1(\alpha_i, \alpha_j) = d_2(\alpha_i, \alpha_j) = 0$ for all $\alpha_i, \alpha_j \in W$ with $\alpha_i \neq \alpha_j$); and

2. $d_1(\alpha_i, \alpha_i) = d_2(\alpha_i, \alpha_i)$ for all $\alpha_i \in W$.

It may readily be checked that this is indeed an equivalence relation on the space $\mathcal{D}$ of all decoherence functions.

Each equivalence class of W-equivalent decoherence functions may be represented by the member $d_{\tilde{X}}$ whose decoherence operator has the 'canonical form' (this decoherence function has been useful in other contexts, see [31])

$$\tilde{X} = \sum_{i=1}^{n} \frac{d(\alpha_i, \alpha_i)}{(\dim \alpha_i)^2} \alpha_i \otimes \alpha_i. \tag{12}$$

We shall shortly be making use of the fact that $\tilde{X}$ is a positive self-adjoint operator on $\mathcal{V} \otimes \mathcal{V}$.

Let us denote the space of W-equivalence classes by $\mathcal{D}_W$. Then $\delta \in \mathcal{D}_W$ is said to be W-*pure* if it cannot be written in the form

$$\delta = \lambda \delta_1 + (1 - \lambda)\delta_2 \tag{13}$$

with $\delta_1 \neq \delta_2$; if δ can be written in the form (13) with $\delta_1 \neq \delta_2$ with $0 \leq \lambda \leq 1,\ \lambda \in \mathbb{R}$, we say it is W-*impure.* Note that the sum of equivalence classes on the right hand side of (13)

[4] Note that in [30] the word window is used to describe the Boolean algebra generated by this set of propositions, rather than the set of propositions itself.

is well-defined since if $d_1 \equiv_W d_1'$ and $d_2 \equiv_W d_2'$ then $\lambda d_1 + (1-\lambda)d_2 \equiv_W \lambda d_1' + (1-\lambda)d_2'$, where $\equiv_W$ means W-equivalent.

Clearly any δ that can be represented by a decoherence operator $\tilde{X}$ of the form (12) with more than one non-zero $d(\alpha,\alpha)$ is impure. On the other hand, consider a decoherence operator

$$\tilde{X} := \frac{1}{(\dim\alpha)^2}\alpha\otimes\alpha \tag{14}$$

where α is one of the members of W and suppose that its associated W-equivalence class of decoherence functions δ can be decomposed in the form (13). Now $\delta_1(\beta,\beta) \geq 0$ and $\delta_2(\beta,\beta) \geq 0$ for all β. Also, since $\delta(\beta,\beta) = 0$ for all $\beta \in W$ such that $\beta \neq \alpha$, we have

$$\delta(\beta,\beta) = 0 = \lambda\delta_1(\beta,\beta) + (1-\lambda)\delta_2(\beta,\beta) \quad \forall\beta \in W \text{ such that } \beta \neq \alpha \tag{15}$$

and therefore

$$0 = \delta_1(\beta,\beta) = \delta_2(\beta,\beta) \quad \forall\beta \in W \text{ such that } \beta \neq \alpha. \tag{16}$$

Also $\delta_1(1,1) = 1$ and $\delta_2(1,1) = 1$, which implies $\delta_1(\sum_{\alpha_i\in W}\alpha_i, \sum_{\alpha_j\in W}\alpha_j) = 1$ which in turn implies $\delta_1(\alpha,\alpha) = 1$, and similarly $\delta_2(\alpha,\alpha) = 1$. Thus δ_1 and δ_2 are both equal to δ and hence δ is pure.

III. INFORMATION-ENTROPY

We turn now to the question of defining the information-entropy in the context of a window and for a given decoherence function. In standard single-time quantum theory the information-entropy is given by

$$I_{s-t} = -\mathrm{tr}(\rho\log\rho) \tag{17}$$

where ρ is the density matrix, and, as mentioned in the Introduction, I_{s-t} is minimised on pure states.

What we seek is a notion of information-entropy that can be used in generalised history quantum theory. In particular, the definition should be applicable in principle to systems in which the concept of time is not fundamental and may emerge only in some coarse-grained way. Furthermore, even if the system has a standard notion of time, the information-entropy—which encodes the number of bits required to describe the system—may not necessarily all reside in the initial state. The description of a system in this generalised type of quantum theory is given entirely in terms of the set of propositions and the values of the decoherence function, so we must construct our measure of information-entropy solely from these.

Firstly, however, we point out that since—as explained above—the decoherence function can be described in terms of a decoherence operator X, the most naïve approach (without physical motivation) might be to try to construct a measure of information-entropy for a decoherence function from X. The simplest analogue of (17) is

$$I_d = -\mathrm{tr}(X \log X) \tag{18}$$

but this is not defined in general since X is neither self-adjoint nor positive.

However, focussing on the probability distributions derived from d does, in fact, offer a way of defining information-entropy. To see this consider a general probability distribution with M events $\{e_i\}_{i=1}^{M}$ with probabilities $\{\mathrm{Prob}(e_i)\}_{i=1}^{M}$. The usual measure of the information-entropy of this distribution is

$$-\sum_{i=1}^{M} \mathrm{Prob}(e_i) \log \mathrm{Prob}(e_i). \tag{19}$$

On the other hand, a given decoherence function produces not one but many probability distributions, namely one for each consistent window. A possible start, therefore, might be to define the information-entropy in the context of a window $W = \{\alpha_i\}$ as

$$\begin{aligned} I_W^{trial} &= -\sum_i \mathrm{Prob}(\alpha_i) \log \mathrm{Prob}(\alpha_i) \\ &= -\sum_i d(\alpha_i, \alpha_i) \log d(\alpha_i, \alpha_i) \end{aligned} \tag{20}$$

which is indeed now well-defined. However, as noted by Hartle [21], I_W^{trial} does not have the appropriate properties with respect to refinement of the consistent set. In particular, we require that if the consistent set is refined—corresponding to having finer-grained propositions—the information-entropy should decrease or stay the same. However (20) does not have this property—indeed, the most coarse grained set $W = \{0, 1\}$ (where 1 is the projector onto the whole Hilbert space) is always consistent and has $d(1, 1) = 1$ and $d(0, 0) = 0$, so that $I_W^{trial} = 0$. As Hartle [21] puts it, there is no penalty for asking stupid questions.

In his very interesting paper [21], Hartle considered this problem and proposed the following definition of 'space-time information-entropy'. First choose a measure of the missing information $S(d)$ in the decoherence function d; for example, one could choose a standard class $\mathcal{C}_{\rm stand}$ of consistent sets and then define $S(d)$ as

$$S(d) := \min_{W \in \mathcal{C}_{\rm stand}} \left[-\sum_\alpha d(\alpha, \alpha) \log \ d(\alpha, \alpha) \right], \tag{21}$$

where W is varied over all consistent sets of histories in the standard class. Hartle suggests that $\mathcal{C}_{\rm stand}$ might be chosen to be the class of *finest* grained histories that decohere.

Having chosen $S(d)$ the next step is to use the Jaynes construction [32] to define the missing information in a general set of decoherent histories W. The missing information is the maximum of the information content of decoherence functions which reproduce the decoherence and probabilities of the set W:

$$S(W, d) := \max_{\tilde{d}} \left[S(\tilde{d}) \right]_{\tilde{d}(\alpha,\alpha) = d(\alpha,\alpha)}, \tag{22}$$

where the maximum is taken over all decoherence functions $\tilde{d}$ that reproduce the decoherence function for the set of histories W. Finally, the missing information in any class $\mathcal{C}$ of decoherent sets of histories is defined as

$$S(\mathcal{C}, d) = \min_{W \in \mathcal{C}} [S(W, d)]. \tag{23}$$

We feel that the definition of information-entropy that we shall now develop has a number of potential advantages over that given by Hartle. In particular (i) it is fairly simple; (ii)

it does not need the use of maximum entropy ideas; (iii) it does not require the choice a standard class of consistent sets. We shall also show in the next section, when calculated in the case of standard quantum theory (for consistent sets of homogeneous histories) the information-entropy for the decoherence function is found to be equal to $-\mathrm{tr}(\rho \log \rho)$, up to normalisation, where ρ is the initial density matrix.

As a first step towards finding this new definition of information-entropy consider any window $W = \{\alpha_i\}_{i=1}^n$ that is consistent with respect to a given decoherence function d. Then, as explained above, the canonical decoherence operator $\tilde{X}_{d,W}$ that reproduces the values of $d(\alpha_i, \alpha_j)$ of d in the window W is

$$\tilde{X}_{d,W} = \sum_{i=1}^{n} \frac{d(\alpha_i, \alpha_i)}{(\dim \alpha_i)^2} \alpha_i \otimes \alpha_i. \tag{24}$$

The crucial observation is that, unlike a general decoherence operator, $\tilde{X}_{d,W}$ *is* a positive, self-adjoint operator on $\mathcal{V} \otimes \mathcal{V}$, and so one can define the logarithm of $\tilde{X}_{d,W}$ and thereby form

$$-\,\mathrm{tr}(\tilde{X}_{d,W} \log \tilde{X}_{d,W}) = -\sum_{i=1}^{n} d(\alpha_i, \alpha_i) \log \left(\frac{d(\alpha_i, \alpha_i)}{(\dim \alpha_i)^2} \right). \tag{25}$$

This motivates the following definition of the information-entropy for this decoherence function and window:

$$\hat{I}_{d,W} := -\sum_{i=1}^{n} d(\alpha_i, \alpha_i) \log \frac{d(\alpha_i, \alpha_i)}{(\dim \alpha_i)^2}. \tag{26}$$

While this function has many of the propeties that are desired of a measure of information, as we shall show below, there are persuasive arguments [5] for renormalising this function and to define our measure of information as

$$I_{d,W} := \hat{I}_{d,W} - \hat{I}_{d,\{1,0\}}$$

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[5]We are extremely grateful to J Hartle and A Kent for reading an earlier draft of this paper and bringing these issues to our attention.

$$= -\left(\sum_{i=1}^{n} d(\alpha_i,\alpha_i)\log\frac{d(\alpha_i,\alpha_i)}{(\dim\alpha_i)^2}\right) - \log\dim\mathcal{V}^2$$

$$= -\sum_{i=1}^{n} d(\alpha_i,\alpha_i)\log\frac{d(\alpha_i,\alpha_i)}{(\dim\alpha_i/\dim\mathcal{V})^2}, \tag{27}$$

where $\hat{I}_{d,\{1,0\}}$ is the value of $\hat{I}_{d,W}$ for the (coarsest) window $\{1,0\}$, and $\mathcal{V}$ is the Hilbert space on which the history propositions are defined.

The function

$$\frac{\dim\alpha_i}{\dim\mathcal{V}} \tag{28}$$

is the *relative* dimension of the projector α_i. Before we go further to describe properties of this measure of information-entropy, let us describe the reasons for this use of relative dimension rather than absolute dimension of a proposition. The difference is clearly only important if one envisages comparing information-entropy in situations where $\mathcal{V}$ changes. An important case in point is the history version of n-time quantum mechanics. In this case, $\mathcal{V} = \otimes^n\mathcal{H}$, where $\mathcal{V} = \otimes^n\mathcal{H}$ is the Hilbert space of the canonical theory. One uses n-times to model a situation in which 'nothing happens' in the intermediate times.

Consider a consistent set of n-time histories

$$\{\alpha_i\} = \{P_1^i, P_2^i ... P_{t_r}^i, P_{t_{r+1}}^i, ... P_n^i\}. \tag{29}$$

Now imagine inserting an additional time, between t_r and t_{r+1}, say, but use the unit projector at this time. The use of relative dimension ensures that the information-entropy does not change when one does this trivial extension to the consistent set, since the dimension of the history

$$\alpha_i = (P_1^i, P_2^i ... P_{t_r}^i, 1, P_{t_{r+1}}^i, ... P_n^i) \tag{30}$$

is $\dim\mathcal{H}$ times that of

$$\alpha_i = (P_1^i, P_2^i ... P_{t_r}^i, P_{t_{r+1}}^i, ... P_n^i) \tag{31}$$

however the dimension of $\mathcal{V}$ in (30) is $\dim\mathcal{H}$ times that of (31).

An additional aspect of the use of relative dimension is that it may help in extending our work to infinite dimensions, since it may be possible to use von Neumann's theory of dimension functions of type II_1 algebras of projectors [33].

Returning now to consideration of general properties of (27), we note that the definition we have given is close to the simple form (20), however the extra factor $(\dim \alpha_i)^{-2}$ is the crucial ingredient that results in the thus-defined information-entropy being either constant or decreasing when the window is refined. To see this, consider two consistent windows $W_1 = \{\alpha, \alpha_1, \alpha_2 \ldots, \alpha_n\}$ and $W_2 = \{\beta, \gamma, \alpha_1, \alpha_2 \ldots, \alpha_n\}$ where W_2 is a refinement of W_1 in the sense that one of the projection operators in W_1, namely α, is split into two with $\alpha = \beta \oplus \gamma$. Thus

$$\begin{aligned} I_{d,W_1} - I_{d,W_2} = -d(\alpha,\alpha)\log\left(\tfrac{d(\alpha,\alpha)}{(\dim\alpha)^2}\right) + d(\beta,\beta)\log\left(\tfrac{d(\beta,\beta)}{(\dim\beta)^2}\right) \\ + d(\gamma,\gamma)\log\left(\tfrac{d(\gamma,\gamma)}{(\dim\gamma)^2}\right). \end{aligned} \tag{32}$$

For simplicity of notation it will be convenient to define the ratios

$$a := \frac{d(\gamma,\gamma)}{d(\beta,\beta)} \text{ and } b := \frac{\dim(\gamma)}{\dim(\beta)} \tag{33}$$

and, without loss of generality, we can take $0 \le a < \infty$ and $1 \le b < \infty$. Now $d(\alpha,\alpha) = d(\beta,\beta) + d(\gamma,\gamma)$ and $\dim(\alpha) = \dim(\beta) + \dim(\gamma)$, and hence

$$I_{d,W_1} - I_{d,W_2} = d(\beta,\beta)\left(a\log\left(\frac{a}{b^2}\right) - (1+a)\log\left(\frac{(1+a)}{(1+b)^2}\right)\right). \tag{34}$$

It is not too difficult to prove the inequality

$$a\log\left(\frac{a}{b^2}\right) - (1+a)\log\left(\frac{(1+a)}{(1+b)^2}\right) \ge 0 \quad \text{for} \quad 0 \le a < \infty \text{ and } 1 \le b < \infty, \tag{35}$$

which implies that I_{d,W_1} decreases with respect to this special type of refinement. However, *any* refinement of W_1 can be reached in a step-wise fashion by repeated refining of one projection operator into two, and hence $I_{d,W}$ decreases under any refinement. We note in particular that with this definition of $I_{d,W}$ the consistent set $W = \{1, 0\}$ has information-entropy 0 and that this is the maximum possible value of the function $I_{d,W}$; the minimum possible value of

$$\hat{I}_{d,W} := -\sum_{i=1}^{n} d(\alpha_i, \alpha_i) \log \frac{d(\alpha_i, \alpha_i)}{(\dim \alpha_i)^2}. \tag{36}$$

for any d and W is zero (which occurs if there is a consistent set all of whose projectors are one-dimensional and all of whose probabilities, bar one, are zero) so that the minimum value of $I_{d,W}$ is $-2 \log \dim \mathcal{V}$.

At this stage we might proceed in several different ways. One possibility is to leave the information-entropy defined in this 'localised' form $I_{d,W}$ in which the context W appears explicitly. This procedure would be rather natural within the topos-theoretic interpretation of the consistent histories formalism that was introduced recently by one of us [30]. In this case it is appropriate to define I_{d,W_0} for *any* window W_0 (*i.e.,* any set of exclusive and exhaustive histories, not necessarily one that is d-consistent) as

$$I_{d,W_0} := \min_{W \geq W_0} I_{d,W} \tag{37}$$

where the minimisation is taken over all coarse-grainings W of W_0 that *are* d-consistent. We note that $W_0 \mapsto I_{d,W_0}$ is an order-preserving map from the partially-ordered set of windows to the ordered set of real numbers—an essentially 'functorial' property in the language of [30].

A second possibility is to define *the* information-entropy of the decoherence function d as the minimum over all consistent sets of $I_{d,W}$, *i.e.,*

$$I_d := \min_W I_{d,W}. \tag{38}$$

As will become clear from the examples below the consistent set (or sets) which minimise $I_{d,W}$ seem to be naturally associated with the decoherence operator X.

Before proceeding to illustrate these ideas with examples drawn from standard n-time quantum theory it is worth emphasising that this definition of I_d is non-trivial; in particular, it is not independent of d (as, *a priori*, it might have been). To demonstrate this point consider the decoherence function associated with the following decoherence operator on the space $\mathcal{V} \otimes \mathcal{V}$ with $\mathcal{V} = \mathbb{C}^2$:

$$X_1 := \frac{1}{2}\left(\alpha \otimes \beta + \beta \otimes \alpha\right) \tag{39}$$

where

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{40}$$

If P is the most general one-dimensional projection operator

$$P = \begin{pmatrix} a & b \\ b^* & 1-a \end{pmatrix} \tag{41}$$

with $a \in \mathbb{R},\ 0 \leq a \leq 1$ and $b \in \mathbb{C},\ |b|^2 = a(1-a)$ one may easily calculate that

$$d(P, 1-P) = \frac{1}{2}\left(a^2 + (1-a)^2\right) \tag{42}$$

so that there are no one-dimensional consistent sets and the only consistent set is $\{0, 1\}$ which has $I_{d_1} = 0$. On the other hand, the decoherence function d_2 associated with the decoherence operator

$$X_2 = \alpha \otimes \alpha \tag{43}$$

on the same space has $I_{d_2} = -2\log 2$ (since in this case the set $\{\alpha, \beta\}$ is consistent and has $I_{d_2} = -2\log 2$).

IV. EXAMPLES

A. The history version of standard quantum theory

The definition we have given for I_d is in terms of consistent sets and their associated probability distributions and it is interesting to see how it reduces up to normalisation to the familiar information-entropy

$$I_{s-t} = -\mathrm{tr}(\rho \log \rho) \tag{44}$$

of standard quantum theory. In this case the histories are simply projectors at one time point and the value of the decoherence function is

$$d(P,Q) = \mathrm{tr}(P\rho Q). \tag{45}$$

Firstly it should be noted that all exhaustive and exclusive sets are consistent since if P_1 and P_2 are two orthogonal projectors then

$$d(P_1,P_2) = \mathrm{tr}(P_1\rho P_2) = \mathrm{tr}(\rho P_2 P_1) = 0. \tag{46}$$

Secondly since—as shown in the previous section—the information-entropy is constant or decreases upon refinement of the window, it suffices to consider windows in which all the projectors are one-dimensional. Thus the information-entropy of the decoherence function is the minimum over all one-dimensional resolutions of the identity $W = \{P_i\}_{i=1}^{N}$ of $I_{d,W}$ (where N is the dimension of the Hilbert space $\mathcal{V}$; of course, in this case, $\mathcal{V}$ is just the canonical Hilbert space $\mathcal{H}$)

$$\begin{aligned} I_{d,W} &= -\sum_i d(P_i,P_i)\log d(P_i,P_i) - 2\log N \\ &= -\sum_i \mathrm{tr}(P_i\rho P_i)\log \mathrm{tr}(P_i\rho P_i) - 2\log N \\ &= -\sum_i \mathrm{tr}(\rho P_i)\log \mathrm{tr}(\rho P_i) - 2\log N. \end{aligned} \tag{47}$$

This expression for $I_{d,W}$ is independent of the basis in which the traces are evaluated and it is convenient to evaluate it in the basis in which the density matrix, ρ, is diagonal:

$$I_{d,W} = -\sum_i \left(\sum_j (P_i)_{jj} r_j\right) \log \left(\sum_j (P_i)_{jj} r_j\right) - 2\log N \tag{48}$$

where the $(P_i)_{jj}$ are the diagonal elements of P_i in this basis and r_j are the (possibly repeated) eigenvalues of ρ.

Now the function $f(x) = -x\log x$ is a concave function and hence satisfies the inequality

$$f\left(\sum_j l_j x_j\right) \geq \sum_j l_j f(x_j) \tag{49}$$

where the positive real numbers l_i satisfy $0 \le l_i \le 1$ and $\sum_i l_i = 1$ (this is essentially Jensen's inequality; see for example [34]).

We shall now use this inequality to get a lower bound on $I_{d,W}$. The one-dimensional projectors have trace 1, *i.e.,*

$$\sum_j (P_i)_{jj} = 1 \text{ for each } i \tag{50}$$

and therefore we can use (49) for each i with $(P_i)_{jj}$ playing the rôle of l_j, so that

$$\begin{aligned} I_{d,W} &= -\sum_i \left(\sum_j (P_i)_{jj} r_j \right) \log \left(\sum_j (P_i)_{jj} r_j \right) - 2 \log N \\ &\ge -\sum_i \left(\sum_j (P_i)_{jj} \left(r_j \log r_j \right) \right) - 2 \log N. \end{aligned} \tag{51}$$

However, since $\sum_i P_i = 1$, in any basis we have

$$\sum_i (P_i)_{jj} = 1 \text{ for each } j \tag{52}$$

thus

$$\begin{aligned} I_{d,W} &\ge -\sum_i \left(\sum_j (P_i)_{jj} \left(r_j \log r_j \right) \right) - 2 \log N \\ &= -\sum_j \left(r_j \log r_j \right) - 2 \log N \\ &= -\mathrm{tr}(\rho \log \rho) - 2 \log N. \end{aligned} \tag{53}$$

We note that, if ρ is non-degenerate, by choosing the P_i to be the spectral projections of ρ we can indeed attain the bound. Hence, if we define the information-entropy of the decoherence function to be the minimum of $I_{d,W}$ over all W we find

$$I_d = -\mathrm{tr}(\rho \log \rho) - 2 \log N. \tag{54}$$

If ρ is degenerate, we should choose a resolution of the identity by one-dimensional projectors obtained by replacing each n-dimensional spectral projector Q by any set of orthogonal projectors which sum to Q; again one finds that $I_d = -\mathrm{tr}(\rho \log \rho) - 2 \log N$.

Thus the definition of information-entropy that we have given reduces, up to normalisation, to the usual one in the case of single-time quantum theory.

B. n-time quantum theory

We recall that in standard n-time quantum theory a history is a time-ordered sequence of propositions about the system $\alpha = (P^1_{t_1}, P^2_{t_2}, \ldots, P^n_{t_n})$ with $t_1 < t_2 < \cdots < t_n$. As we have argued in [18,19], this proposition should be associated with the operator $(P^1_{t_1} \otimes P^2_{t_2} \otimes \cdots \otimes P^n_{t_n})$ on the tensor product Hilbert space $\mathcal{V} = \mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2} \otimes \cdots \otimes \mathcal{H}_{t_n}$ of n copies of the Hilbert space $\mathcal{H}$ on which the canonical theory is defined. We have called histories such as $(P^1_{t_1} \otimes P^2_{t_2} \otimes \cdots \otimes P^n_{t_n})$, represented by a tensor product of operators on $\mathcal{V}$, *homogeneous*; there are, of course, many projectors on $\mathcal{V}$ which are not of this form. In [20] we have shown how to construct the operator X (an operator on $\mathcal{V} \otimes \mathcal{V}$) in this case so as to reproduce the standard expression for the decoherence function, namely

$$d(\alpha, \beta) = tr_{\mathcal{H}}(\tilde{C}^{\dagger}_{\alpha} \rho_{t_0} \tilde{C}_{\beta}) \tag{55}$$

where

$$\tilde{C}_{\alpha} = U(t_0, t_1) P^1_{t_1} U(t_1, t_2) P^2_{t_2} U(t_2, t_3) \ldots U(t_{n-1}, t_n) P^n_{t_n} U(t_n, t_0) \tag{56}$$

and $U(t, t') = e^{-i(t-t')H}$ is the usual time-evolution operator in the Hilbert space $\mathcal{H}$ of the canonical theory.

We now show that the minimum value of the information-entropy $I_{d,W}$ over all consistent sets of homogeneous histories for standard quantum theory is

$$-\operatorname{tr}(\rho \log \rho) - 2 \log \dim \mathcal{V}. \tag{57}$$

We suspect that this value is the minimum for *any* consistent sets (*i.e.,* including inhomogeneous histories) but so far we have only been able to prove this in certain examples (see below).

Firstly we note that by taking the projection operators at each time to be related to the spectral projectors of ρ we can find a consistent set that gives the value (57) as the information-entropy of that set. More precisely, choose the histories to be of the form

$$\alpha = (U(t_0,t_1)^{-1}P^i_{t_1}U(t_1,t_0)^{-1}, U(t_0,t_2)^{-1}P^j_{t_2}U(t_2,t_0)^{-1}, \ldots,$$
$$\ldots, U(t_0,t_n)^{-1}P^n_{t_n}U(t_n,t_0)^{-1}) \tag{58}$$

where P_i is a one-dimensional projector onto the ith eigenspace of ρ. The unitary operators are needed to 'undo' the time evolution so that expressions for the probabilities become of the form

$$\mathrm{tr}(P_n \ldots P_j P_i \rho P_i P_j \ldots P_n). \tag{59}$$

It is easy to see that all of the histories so defined will have zero probability except those of the form $(P_i, P_i, \ldots, P_i)$, $i = 1, 2, \ldots, N$ (where N is the dimension of the Hilbert space) which have probabilities

$$\mathrm{tr}(P_i \ldots P_i P_i \rho P_i P_i \ldots P_i) = \mathrm{tr}(\rho P_i) = r_i \tag{60}$$

where r_i are the eigenvalues of ρ. Thus the value of the information-entropy in this window is

$$I_{d,W} = -\sum_i^N r_i \log r_i - 2\log\dim\mathcal{V} = -\mathrm{tr}(\rho\log\rho) - 2\log\dim\mathcal{V}. \tag{61}$$

In order show that the value of $I_{d,W}$ in any other window of homogeneous projectors is greater than this it is helpful to note the following. Let $\{Q^j\}_{j=1}^{N_1}$ be a resolution of the identity by projectors in the Hilbert space $\mathcal{H}$, and let K be a positive self-adjoint operator. Then

$$-\sum_{j=1}^{N_1}\mathrm{tr}(Q^jKQ^j)\log\mathrm{tr}\left(\frac{(Q^jKQ^j)}{(\dim Q^j)^2}\right) = -\sum_{j=1}^{N_1}\mathrm{tr}(Q^jK)\log\mathrm{tr}\left(\frac{(Q^jK)}{(\dim Q^j)^2}\right)$$
$$\geq -\mathrm{tr}(K\log K), \tag{62}$$

and also

$$-\sum_{j=1}^{N_1}\mathrm{tr}\left((Q^jKQ^j)\log\left(\frac{(Q^jKQ^j)}{(\dim Q^j)^2}\right)\right) \geq -\mathrm{tr}(K\log K). \tag{63}$$

The inequality (62) is essentially the same as that proven in the previous subsection (it should be noted that that proof did not depend on the fact that ρ had trace 1). The

inequality (63) may most easily be seen by considering the left hand side in a basis in which the Q^j's are simultaneously diagonal. Then the operator Q^jKQ^j is of the block diagonal form

$$Q^jKQ^j = \begin{pmatrix} (0) & & & & \\ & (0) & & & \\ & & (K^j) & & \\ & & & (0) & \\ & & & & (0) \end{pmatrix} \tag{64}$$

where K^j is a $\dim Q^j \times \dim Q^j$ positive self-adjoint matrix. Clearly

$$\begin{aligned} -\operatorname{tr}_{N\times N}\left(Q^jKQ^j \log \frac{Q^jKQ^j}{(\dim Q^j)^2}\right) &= -\operatorname{tr}_{\dim Q^j\times \dim Q^j}\left(K^j \log \frac{K^j}{(\dim Q^j)^2}\right) \\ &\geq -\operatorname{tr}_{\dim Q^j\times \dim Q^j}\left(K^j \log K^j\right). \end{aligned} \tag{65}$$

We may now use these results and one from the previous section to find an upper bound on the information-entropy for any consistent window of homogeneous projectors. We have

$$\begin{aligned} I_{d,W} = &- \sum_{i_1,i_2,\ldots,i_{n-1},i_n} \operatorname{tr}(P_{i_n}P_{i_{n-1}}\ldots P_{i_2}P_{i_1}\rho P_{i_1}P_{i_2}\ldots P_{i_{n-1}}P_{i_n}) \\ &\quad \times \log\operatorname{tr}\left(\frac{(P_{i_n}P_{i_{n-1}}\ldots P_{i_2}P_{i_1}\rho P_{i_1}P_{i_2}\ldots P_{i_{n-1}}P_{i_n})}{(\dim P_{i_1}\dim P_{i_2}\ldots\dim P_{i_{n-1}}\dim P_{i_n})^2}\right) - 2\log\dim\mathcal{V} \\ = &- \sum_{i_1,\ldots,i_{n-1}} (\dim P_{i_1}\ldots\dim P_{i_{n-1}})^2 \\ &\quad \times \sum_{i_n} \operatorname{tr}\left(P_{i_n}\left[\frac{P_{i_{n-1}}\ldots P_{i_1}\rho P_{i_1}\ldots P_{i_{n-1}}}{(\dim P_{i_1}\ldots\dim P_{i_{n-1}})^2}\right]\right) \\ &\quad \times \log\operatorname{tr}\left(\frac{P_{i_n}}{(\dim P_{i_n})^2}\left[\frac{P_{i_{n-1}}\ldots P_{i_1}\rho P_{i_1}\ldots P_{i_{n-1}}}{(\dim P_{i_1}\ldots\dim P_{i_{n-1}})^2}\right]\right) - 2\log\dim\mathcal{V} \end{aligned} \tag{66}$$

and hence (62) can be used to show that

$$\begin{aligned} &I_{d,W} \\ &\geq - \sum_{i_1,i_2,\ldots,i_{n-1}} (\dim P_{i_1}\ldots\dim P_{i_{n-1}})^2 \\ &\quad\times\quad \operatorname{tr}\left(\left[\frac{P_{i_{n-1}}\ldots P_{i_1}\rho P_{i_1}\ldots P_{i_{n-1}}}{(\dim P_{i_1}\ldots\dim P_{i_{n-1}})^2}\right]\log\left[\frac{P_{i_{n-1}}\ldots P_{i_1}\rho P_{i_1}\ldots P_{i_{n-1}}}{(\dim P_{i_1}\ldots\dim P_{i_{n-1}})^2}\right]\right) \end{aligned}$$

$$
\begin{aligned}
&-2\log\dim\mathcal{V} \\
&= -\sum_{i_1,i_2,\ldots,i_{n-1}} \operatorname{tr}\left(\left[P_{i_{n-1}}\ldots P_{i_1}\rho P_{i_1}\ldots P_{i_{n-1}}\right]\log\left[\frac{P_{i_{n-1}}\ldots P_{i_1}\rho P_{i_1}\ldots P_{i_{n-1}}}{(\dim P_{i_1}\ldots\dim P_{i_{n-1}})^2}\right]\right) \\
&\quad -2\log\dim\mathcal{V}.
\end{aligned} \tag{67}
$$

We may now use (63) to give

$$
\begin{aligned}
&I_{d,W} \\
&\geq -\sum_{i_1,i_2,\ldots,i_{n-1}} \operatorname{tr}\left(\left[P_{i_{n-1}}\ldots P_{i_1}\rho P_{i_1}\ldots P_{i_{n-1}}\right]\log\left[\frac{P_{i_{n-1}}\ldots P_{i_1}\rho P_{i_1}\ldots P_{i_{n-1}}}{(\dim P_{i_1}\ldots\dim P_{i_{n-1}})^2}\right]\right) - 2\log\dim\mathcal{V} \\
&= -\sum_{i_1,i_2,\ldots,i_{n-2}} (\dim P_{i_1}\ldots\dim P_{i_{n-2}})^2 \\
&\quad\times\sum_{i_{n-1}} \operatorname{tr}\Bigg(P_{i_{n-1}}\left[\frac{P_{i_{n-2}}\ldots P_{i_1}\rho P_{i_1}\ldots P_{i_{n-2}}}{(\dim P_{i_1}\ldots\dim P_{i_{n-2}})^2}\right]P_{i_{n-1}} \\
&\qquad\times\log\frac{P_{i_{n-1}}}{(\dim P_{i_{n-1}})^2}\left[\frac{P_{i_{n-2}}\ldots P_{i_1}\rho P_{i_1}\ldots P_{i_{n-2}}}{(\dim P_{i_1}\ldots\dim P_{i_{n-2}})^2}\right]P_{i_{n-1}}\Bigg) - 2\log\dim\mathcal{V} \\
&\geq -\sum_{i_1,i_2,\ldots,i_{n-2}} \operatorname{tr}\left(\left[P_{i_{n-2}}\ldots P_{i_1}\rho P_{i_1}\ldots P_{i_{n-2}}\right]\log\left[\frac{P_{i_{n-2}}\ldots P_{i_1}\rho P_{i_1}\ldots P_{i_{n-2}}}{(\dim P_{i_1}\ldots\dim P_{i_{n-2}})^2}\right]\right) - 2\log\dim\mathcal{V}.
\end{aligned} \tag{68}
$$

The right-hand-side of this expression is of the same form as the right-hand-side of (67) but with one less summation ($n-2$ summations compared to $n-1$ in (67)). We may now repeat this step recursively to show that

$$
I_{d,W} \geq \operatorname{tr}(\rho\log\rho) - 2\log\dim\mathcal{V}. \tag{69}
$$

Thus the minimum of $I_{d,W}$ over consistent windows containing only homogeneous histories is $-\operatorname{tr}(\rho\log\rho) - 2\log\dim\mathcal{V}$. In other words, all the information-entropy lies in the initial state for standard n time quantum theory with unitary evolution.

It is worth noting that if the time evolution is *non*-unitary (such as might occur in a space-time region around a black hole) then histories such as (58) cannot be used to minimise the information-entropy since the operators at each time are no longer projectors. Thus there will be a contribution to the information-entropy from the time evolution in addition to that from the initial state.

C. Two-time histories

As was remarked earlier, we suspect that $-\mathrm{tr}(\rho\log\rho) - 2\log\dim\mathcal{V}$ is the minimum of $I_{d,W}$ over all consistent sets although, so far, we have only been able to show this in certain special cases. One such is the two-time history version of a quantum system with canonical Hilbert space $\mathcal{H} = \mathbb{C}^N$ (*i.e.,* the history Hilbert space is $\mathcal{V} = \mathcal{H}\otimes\mathcal{H} = \mathbb{C}^{N^2}$) with a unitary time evolution and the special initial density matrix

$$\rho = \mathrm{diag}(\frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N}). \tag{70}$$

In fact we may take the time evolution to be trivial (*i.e.* we choose the Hamiltonian to be zero) without loss of generality, since the class of history propositions we will consider takes into account all possible unitary evolutions.

We now show that the minimum of $I_{d,W}$ over *all* consistent sets is

$$-\mathrm{tr}_{\mathcal{H}}(\rho\log\rho) - 2\log\dim\mathcal{V} = \log N - 2\log N^2 = -3\log N. \tag{71}$$

The most general unit vector in the tensor product space $\mathcal{V} = \mathcal{H}\otimes\mathcal{H}$ is

$$v = \sum_{i,j=1}^{N} v^{ij}|i\rangle\otimes|j\rangle \tag{72}$$

where $\{|i\rangle\}_{i=1}^{N}$ is an orthonormal basis for $\mathcal{H}$ and the constants v^{ij} satisfy

$$\sum_{ij} v^{ij}v^{*ij} = 1. \tag{73}$$

The one-dimensional projection operator onto the subspace defined by this vector is

$$\begin{aligned} P_v &= \sum_{ijkm} \left(v^{ij}|i\rangle\otimes|j\rangle\right)\left(v^{*km}\langle k|\otimes\langle m|\right) \\ &= \sum_{ijkm} v^{ij}v^{*km}\ |i\rangle\langle k|\otimes|j\rangle\langle m|. \end{aligned} \tag{74}$$

If we consider consistent sets that contain only one-dimensional projectors then in order to decrease the information-entropy at least one projector must have a probability greater than $1/N$. However, we will now show that the maximum value of the probability of any

one-dimensional projector in a consistent set is $1/N$, so that no consistent windows with one-dimensional histories reduces the information-entropy below $\log N - 2\log N^2 = -3\log N$, the value obtained by considering windows with only homogeneous histories. We also show that (as might be expected) windows with higher-dimensional histories also fail to reduce the information-entropy below this value.

If P_v is part of a consistent set then

$$d(1 - P_v, P_v) = 0 \tag{75}$$

so that

$$d(1, P_v) = d(P_v + (1 - P_v), P_v) = d(P_v, P_v) + d(1 - P_v, P_v) = d(P_v, P_v) \tag{76}$$

where we have used the fact that if α and β are two disjoint histories, then for any other history γ,

$$d(\alpha \oplus \beta, \gamma) = d(\alpha, \gamma) + d(\beta, \gamma). \tag{77}$$

Thus the probability of this particular consistent history proposition is

$$\begin{aligned} d(P_v, P_v) &= d(1, P_v) \\ &= \mathrm{tr}_{\mathcal{V}\otimes\mathcal{V}}([1 \otimes P_v]\ X) \end{aligned} \tag{78}$$

where X is the decoherence operator for this system which may be found in [20] (as a special case of the results given there):

$$X = [R_{(2)} \otimes 1_2] S_4 [1_2 \otimes (\rho \otimes 1_1)][R_{(2)} \otimes 1_2]. \tag{79}$$

In this expression, 1_1 is the unit operator on $\mathcal{H}$ and 1_2 is the unit operator on $\mathcal{V} = \mathcal{H} \otimes \mathcal{H}$; $R_{(2)}$ is the 'time-reversal' operator on $\mathcal{V} = \mathcal{H} \otimes \mathcal{H}$:

$$R_{(2)} u_1 \otimes u_2 = u_2 \otimes u_1; \quad R_{(2)}^2 = 1; \tag{80}$$

and S_4 is the map on $\otimes^4 \mathcal{H}$ which acts as

$$S_4(u_1 \otimes u_2 \otimes u_3 \otimes u_4) = u_2 \otimes u_3 \otimes u_4 \otimes u_1, \tag{81}$$

which has the important property that, for any four operators A, B, C, D on $\mathcal{H}$,

$$\mathrm{tr}_{\otimes^4\mathcal{H}}([A \otimes B \otimes C \otimes D]\ S_4) = \mathrm{tr}_{\mathcal{H}}(ABCD). \tag{82}$$

Thus,

$$\begin{aligned}
d(1, P_v) &= \mathrm{tr}_{\mathcal{V}\otimes\mathcal{V}}([1_2 \otimes P_v]\ X) \\
&= \mathrm{tr}_{\otimes^4\mathcal{H}}([1_2 \otimes P_v][R_{(2)} \otimes 1_2]S_4[1_2 \otimes (\rho \otimes 1_1)][R_{(2)} \otimes 1_2]) \\
&= \mathrm{tr}_{\otimes^4\mathcal{H}}([R_{(2)} \otimes 1_2][1_2 \otimes P_v][R_{(2)} \otimes 1_2]S_4[1_2 \otimes (\rho \otimes 1_1)]) \\
&= \mathrm{tr}_{\otimes^4\mathcal{H}}([1_2 \otimes P_v]S_4[1_2 \otimes (\rho \otimes 1_1)]) \\
&= \mathrm{tr}_{\otimes^4\mathcal{H}}(S_4[1_2 \otimes \{(\rho \otimes 1_1)P_v\}]) \\
&= \sum_{ijkm} v^{ij} v^{*km} \mathrm{tr}_{\otimes^4\mathcal{H}}(S_4[1_1 \otimes 1_1 \otimes \rho|i\rangle\langle k| \otimes |j\rangle\langle m|]) \\
&= \sum_{ijkm} v^{ij} v^{*km} \mathrm{tr}_{\mathcal{H}}[\rho|i\rangle\langle k|j\rangle\langle m|]
\end{aligned} \tag{83}$$

where we have used (82).

Thus

$$\begin{aligned}
d(1, P_v) &= \frac{1}{N} \sum_{ijkm} v^{ij} v^{*km} \delta^{im} \delta^{kj} \\
&= \frac{1}{N} \sum_{ij} v^{ij} v^{*ji} \\
&= \frac{1}{N} \left(\sum_i v^{ii} v^{*ii} + \sum_{i \neq j} v^{ij} v^{*ji} \right) \\
&= \frac{1}{N} \left(\sum_i v^{ii} v^{*ii} + \sum_{i<j} \left(-|v^{ij} - v^{ji}|^2 + v^{ij} v^{*ij} + v^{ji} v^{*ji} \right) \right) \\
&= \frac{1}{N} \left(1 - \sum_{i<j} |v^{ij} - v^{ji}|^2 \right) \\
&\leq \frac{1}{N}.
\end{aligned} \tag{84}$$

We also note that this calculation shows that for a k dimensional projector $P_{(k)}$, the probability

$$d(P_{(k)}, P_{(k)}) = d(P_{(k)}, 1) \leq \frac{k}{N} \tag{85}$$

so that including higher dimensional projectors in a window will only increase the value of the information-entropy. Hence, for this example, $-\mathrm{tr}_{\mathcal{H}}(\rho \log \rho) - 2 \log \dim \mathcal{V} = -3 \log N$ is the minimum value of $I_{d,W}$ over all windows.

V. CONCLUSION

We have shown that there are no pure decoherence functions in the consistent histories approach to generalised quantum theory since every decoherence function may be written as the sum of two others.

More substantially, we have also put forward a definition of information-entropy in generalised quantum mechanics that relies crucially on the notion of the dimension of a history, a concept that is natural within our approach to the general scheme. It is worth noting that fundamental to the consistent histories approach from the start has been the idea of taking the sum of two homogeneous histories in standard n-time quantum theory to form inhomogeneous histories. However, the idea of the dimension of an inhomogeneous history is difficult to understand unless, as we have frequently advocated, histories are identified with projection operators on an n-fold tensor product space.

We have called the function $I_{d,W}$ a measure of information-entropy for generalised quantum mechanics as it has key properties that it decreases under refinement and it is small for consistent windows in which the probability is peaked around histories of small dimension (as we have shown, decoherence functions may or may not have such windows). The fact that $I_{d,W}$ is negative, however means that it is not quite a usual measure of missing information. One does, of course, have the option of using the negative of the function $I_{d,W}$, however we have not done so in order to facilitate comparison with other approaches.

It ought to be said at this stage that while the function $I_{d,W}$ has many of the properties that one requires of a measure of information-entropy in the space-time context, its true

meaning is still somewhat unclear. In this context it should be noted that any function of the form

$$I_{d,W}^{x} = -\sum_{i=1}^{n} d(\alpha_i, \alpha_i) \log \left[\frac{d(\alpha_i, \alpha_i)}{(\dim \alpha_i / \dim \mathcal{V})^{x}} \right] \tag{86}$$

where $x \geq 1$ is a real number also has the key property that it decreases under refinement of the consistent set. The case $x = 1$ may turn out to be the most interesting, as in this case, the measure of information is (minus) the Kullback information [35] of the distribution $\{d(\alpha_i, \alpha_i)\}$ relative to a 'maximally ignorant' distribution on the set $\{\alpha_i\}$ which has $\mathrm{Prob}(\alpha_i) = \dim(\alpha_i)/\dim\mathcal{V}$. The relationship between the measures with different values of x needs to be understood. Interestingly, Gell-Mann and Hartle [15] have considered measures of this sort as a result of rather different considerations such as thermodynamic depth [36]. We understand [37] that they have also considered a measure of entropy which they call a 'bundle of histories entropy' which takes into account the number of fine-grained histories in a coarse-grained history; this idea is clearly related to the one we have put forward.

We anticipate that our definition of information-entropy—which is a straightforward function on the class of consistent sets with attractive properties under refinement—may help in the development of a set selection criterion[6]: for example, in the case that the system naturally divides into a subsystem and the 'environment', this might be done by selecting the set which minimises the information-entropy of the distinguished subsystem (see for example [39]). This is an important problem to which we intend to return in future work. Related issues that need to be understood are the rôle of symmetries (see for example [29,40,41]) and the existence of quasi-classical domains and their relation to the system-environment split. In the context of the latter, it should be noted that if our vector space $\mathcal{V}$ happens to arise as the tensor product of two spaces $\mathcal{V}_1$ and $\mathcal{V}_2$, then our definition of information-entropy has precisely the behaviour that might be hoped for. For if one considers a consistent window

[6]The importance of this issue for the whole framework has been discussed by Dowker and Kent [38].

in which each history proposition α is a tensor product $\alpha = \alpha_1 \otimes \alpha_2$, with $\alpha_1 \in P(\mathcal{V}_1)$ and $\alpha_2 \in P(\mathcal{V}_2)$, then the information-entropy is the sum of the information-entropy associated to each sub-system.

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REFERENCES

[1] M. Gell-Mann and J. Hartle, in *Proceedings of the Third International Symposium on the Foundations of Quantum Mechanics in the Light of New Technology*, edited by S. Kobayashi, H. Ezawa, Y. Murayama, and S. Nomura, (Physical Society of Japan, Tokyo, 1990), p321.

[2] M. Gell-Mann and J. Hartle, in *Complexity, Entropy and the Physics of Information, SFI Studies in the Science of Complexity, Vol. VIII*, edited by W. Zurek (Addison-Wesley, Reading, 1990), p425.

[3] M. Gell-Mann and J. Hartle, in *Proceedings of the 25th International Conference on High Energy Physics, Singapore, August, 2–8, 1990*, edited by K.K. Phua and Y. Yamaguchi, (World Scientific, Singapore, 1990).

[4] J. Hartle, in *Quantum Cosmology and Baby Universes*, edited by S. Coleman, J. Hartle, T. Piran, and S. Weinberg, (World Scientific, Singapore, 1991).

[5] J. Hartle, Phys. Rev., D**44**, 3173 (1991).

[6] M. Gell-Mann and J. Hartle, UCSB preprint UCSBTH-91-15 (unpublished).

[7] J. Hartle, in *Proceedings on the 1992 Les Houches School, Gravitation and Quantisation.* 1993.

[8] R.B. Griffiths, J. Stat. Phys. **36**, 219 (1984).

[9] R. Omnès, J. Stat. Phys. **53**, 893 (1988).

[10] R. Omnès, J. Stat. Phys. **53**, 933 (1988).

[11] R. Omnès, J. Stat. Phys. **53**, 957 (1988).

[12] R. Omnès, J. Stat. Phys. **57**, 357 (1989).

[13] R. Omnès, Ann. Phys. (NY) **201**, 354 (1990).

[14] R. Omnès, Rev. Mod. Phys. **64**, 339 (1992).

[15] M. Gell-Mann and J. Hartle, quant-ph/9509054.

[16] R. Griffiths, quant-ph/9505009 and quant-ph/9606004.

[17] R. Omnes, *The Interpretation of Quantum Mechanics*, (Princeton University Press, Princeton, 1994).

[18] C.J. Isham, J. Math. Phys. **23**, 2157 (1994).

[19] C.J. Isham and N. Linden, J. Math. Phys. **35**, 5452 (1994).

[20] C.J. Isham, N. Linden and S. Schreckenberg, J. Math. Phys. **35**, 6360 (1994).

[21] J. Hartle, Phys. Rev. D **51**, 1800 (1995).

[22] J.J. Halliwell, Phys. Rev. D **48**, 2739 (1993).

[23] A. Kent, gr-qc/9610075.

[24] J. McElwaine, quant-ph/9611054.

[25] D.J. Foulis, R.J. Greechie, and G.T. Rüttimann, Int. J. Theor. Phys. **31**, 789 (1992).

[26] V.S. Varadarajan, *The geometry of quantum theory.* (Van Nostrand, New York, 1968).

[27] J.D.M. Wright, J. Math. Phys. **36**, 5409 (1995).

[28] O. Rudolph, gr-qc/9608066, 1996.

[29] S. Schreckenberg, J. Math. Phys (to be published).

[30] C.J. Isham, gr-qc/9607069, 1996, Int. J. Theor. Phys. (to be published).

[31] S. Schreckenberg, J. Math. Phys. **36**, 4735 (1995).

[32] E.T. Jaynes, *Papers on probability, statistics and statistical mechanics.* Edited by R.D Rosenkrantz (Riedel, Dordrecht, 1983).

[33] F.J. Murray and J. von Neumann, Ann. Math. **37**, 116 (1936).

[34] T.M. Cover and J.A. Thomas, *Elements of Information Theory*. (Wiley, New York, 1991).

[35] S. Kullback, *Information Theory and Statistics* (Wiley, New York, 1959).

[36] S. Lloyd and H. Pagels, Ann. Phys. **188**, 186 (1988)

[37] J. Hartle, private communication.

[38] F. Dowker and A. Kent, J. Stat. Phys **82**, 3038 (1995)

[39] W.H. Zurek, in *Physical origins of time asymmetry*, edited by J.J. Halliwell, J. Pérez-Mercader, and W.H. Zurek, (Cambridge, 1994).

[40] S. Schreckenberg, *Imperial College preprint*, Imperial/TP/95-95/49.

[41] T.A. Brun and J.J. Halliwell, Phys. Rev. D., **54**, 2899 (1996).